JOURNAL OF COMPUTATIONAL PHYSICS 41, 136–141 (1981)

A Note on Solving the Buckley–Leverett Equation in the Presence of Gravity

WLODZIMIERZ PROSKUROWSKI

Department of Mathematics, University of Southern California, Los Angeles, California 90007

Received August 26, 1980

The numerical solution to the Buckley–Leverett equation in the presence of gravity is considered. An extension of the use of the Uniform Sampling Method to cases when the fractional flow function has more than one inflection point is described. Results of numerical experiments are presented.

1. INTRODUCTION

In Concus and Proskurowski [1] the numerical solution of the Buckley–Leverett equation in the absence of capillary pressure and gravitational forces was considered. In this note we extend the use of the Uniform Sampling Method (USM) to cases when the fractional flow function $f(u)$ has more than one inflection point. For a description of the USM see Concus and Proskurowski [1, and references therein], where it was called a Random Choice Method. Here we shall only use the property that the USM is an optimally stable scheme, i.e., it is stable whenever the Courant–Friedrichs–Lewy (CFL) condition is met; see Strang [6].

2. SOLUTION OF A RIEMANN PROBLEM

In the USM one is required to solve a set of Riemann problems. The Riemann problem is a hyperbolic differential equation with two constant states u_{L} and u_{R} to the left and to the right of a discontinuity as the initial conditions:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0; \tag{1}$$

$$\begin{aligned} u(x,0) &= u_{\mathrm{L}}, \qquad x \leqslant 0, \\ &= u_{\mathrm{R}}, \qquad x > 0. \end{aligned} \tag{2}$$

Now we present in some detail the solution of the Riemann problem without the

0021-9991/81/050136–06$02.00/0

restrictions given in Concus and Proskurowski [1]; see also the discussion in Gelfand [4, Sects. 8, 9]. Let $f(u)$ in (1) be twice continuously differentiable and $S_{I,J}$ denote the slope of the chord $l_{I,J}(u)$ joining $(u_I, f(u_I))$ and $(u_J, f(u_J))$ for any distinct points u_I and u_J:

$$S_{I,J} = \frac{f(u_I) - f(u_J)}{u_I - u_J}.$$

Denote also

$$a(u) = df(u)/du.$$

On basis of the Rankine–Hugoniot jump condition and the E-condition of Oleinik we obtain the solution to our problem in the following form:

(I) If $u_R < u_L$ then

(1.1) If over $[u_R, u_L]$ the graph of $f(u)$ lies below the chord $l_{L,R}(u)$ then the relation $a(u_R) < S_{L,R} < a(u_L)$ holds over this interval and the state $u = u_L$ is connected to $u = u_R$ by a shock propagating with the speed $S_{L,R}$.

(1.2) If over $[u_R, u_L]$ the graph of $f(u)$ lies above the chord $l_{L,R}(u)$ or the chord $l_{L,R}(u)$ cuts the graph of $f(u)$ then one must construct a convex hull $H(u)$ such that the graph of $f(u)$ remains fully below or on $H(u)$ over this interval. In this case one needs to find all the points u_M in (u_R, u_L) that are the ends of the subintervals on which $H(u)$ and $f(u)$ coincide. We obtain a set of intermediate points $\{u_I\}_0^{n+1}$ ordered as follows: $u_R < u_{M1} < \cdots < u_{Mn} < u_L$. If for any J the graph of $f(u)$ over $[u_J, u_{J+1}]$ lies below the chord $l_{J,J+1}(u)$ then, as in (1.1), u_J is connected to u_{J+1} by a shock propagating with the speed $S_{J,J+1}$. Otherwise, i.e., if over $[u_I, u_{I+1}]$ the hull $H(u)$ coincides with the function $f(u)$, the states u_I and u_{I+1} are connected by an expansion wave.

(II) If $u_L < u_R$ then

(2.1) If over $[u_L, u_R]$ the graph of $f(u)$ lies above the chord $l_{L,R}(u)$ then the relation $a(u_R) < S_{L,R} < a(u_L)$ holds over this interval and the state $u = u_L$ is connected to $u = u_R$ by a shock propagating with the speed $S_{L,R}$.

(2.2) If over $[u_L, u_R]$ the graph of $f(u)$ lies below the chord $l_{L,R}(u)$ or the chord $l_{L,R}(u)$ cuts the graph of $f(u)$ then one must construct a convex hull $h(u)$ such that the graph of $f(u)$ remains fully above or on $h(u)$ over this interval. In this case one needs to find all the points u_m in (u_L, u_R) that are the ends of the subintervals on which $h(u)$ and $f(u)$ coincide. We obtain a set of intermediate points $\{u_I\}_0^{n+1}$ ordered as follows: $u_L < u_{m1} < \cdots < u_{mn} < u_R$. If for any J the graph of $f(u)$ over $[u_J, u_{J+1}]$ lies above the chord $l_{J,J+1}(u)$ then, as in (2.1), u_J is connected to u_{J+1} by a shock propagating with the speed $S_{J,J+1}$. Otherwise, i.e., if over $[u_I, u_{I+1}]$ the hull $h(u)$ coincides with the function $f(u)$, the states u_I and u_{I+1} are connected by an expansion wave.

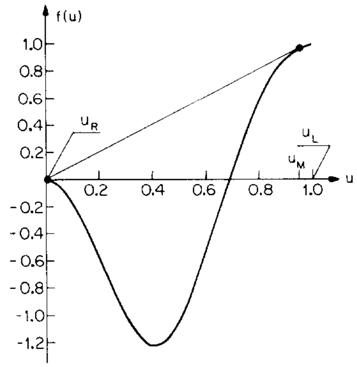

FIG. 1. The shape of the fractional flow function $f(u)$ in the presence of gravity. The case when $u_L > u_R$ is indicated.

3. Examples

An example of (1.2) for the particular choice of $f(u)$ arising in our application is depicted in Fig. 1. There the state $u = u_L$ is connected to $u = u_M$ by an expansion wave, and $u = u_M$ is connected to u_R by a shock propagating with speed $S_{M,R} > 0$. Note that this case does not differ quantitatively from the example shown in Fig. 1 in Concus and Proskurowski [1]. A similar example of (2.2) is depicted in Fig. 2. There the state $u = u_L$ is connected to $u = u_{m1}$ by a shock propagating with speed $S_{L,m1} < 0$, $u = u_{m1}$ is connected to $u = u_{m2}$ by an expansion wave, and $u = u_{m2}$ is connected to $u = u_R$ by a shock propagating with speed $S_{m2,R} > 0$. As a result, two shocks propagating in opposite directions are created, separated by an expansion wave.

In the absence of capillarity, the one dimensional Buckley–Leverett equation for incompressible flow can be written in the form of Eq. (1), in consistant units, with

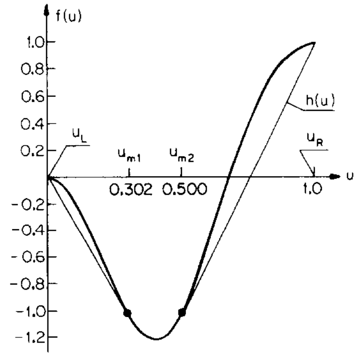

FIG. 2. Construction of a convex hull $h(u)$ for the particular choice of $f(u)$ and the case when $u_L < u_R$.

proper boundary and initial conditions. We consider the flow of oil and water through sand and denote by u the water saturation in the sand. The fractional flow function is $f(u) = f_w(u)(1 - \lambda k_0(u))$, where $f_w(u) = (1 + \alpha k_0(u)/k_w(u))^{-1}$, water relative permeability $k_w(u) = u^2$, oil relative permeability $k_0(u) = (1-u)^2$, α is a constant ratio of viscosities of water and oil, and λ is a constant that includes Darcy velocity, oil viscosity and water–oil density difference. The term $\lambda k_0(u)$ takes into account the gravitational forces; see Douglas *et al.* [2], Fayers and Sheldon [3] and references therein. In our experiments we have used the values of $\alpha = 0.5$ and $\lambda = 10$. The shape of $f(u)$ is depicted in Fig. 1 and we note that it has two inflection points, at $u = 0.208$ and $u = 0.635$.

Equation (1) with the initial conditions

$$\begin{aligned} u(x,0) &= 1 \quad \text{for} \quad x \leqslant 0 \\ &= 0 \quad \text{for} \quad x > 0, \end{aligned} \tag{a}$$

and

$$\begin{aligned} u(x,0) &= 0 \quad \text{for} \quad x \leqslant 0 \\ &= 1 \quad \text{for} \quad x > 0, \end{aligned} \tag{b}$$

can be solved explicitly as a Riemann problem.

In the case of (a) the solution consists of a discontinuity propagating with constant speed, followed by an expansion wave. We have

$$\begin{aligned} u(x,t) &= 0 && \text{for} \quad x > st \\ &= 0.951 && \text{for} \quad x = st \\ &= \text{expansion wave} && \text{for} \quad 0 \leqslant x < st, \end{aligned}$$

where the propagation speed $s = 1.025$. In comparison with the case $\lambda = 0$ (zero gravitational forces) the discontinuity is much steeper, 0.95 vs 0.58, although its speed of propagation is somewhat slower, 1.02 vs 1.37. In the case of (b) the solution consists of two discontinuities propagating in opposite directions with constant speeds, separated by an expansion wave. We have

$$\begin{aligned} u(x,t) &= 0.000 && \text{for} \quad x < s_- t \\ &= 0.302 && \text{for} \quad x = s_- t \\ &= \text{expansion wave} && \text{for} \quad s_- t < x < s_+ t \\ &= 0.500 && \text{for} \quad x = s_+ t \\ &= 1.000 && \text{for} \quad x > s_+ t \end{aligned}$$

where the propagation speed $s_- = -3.49$ and $s_+ = 4.00$. It should be noted that the backward shock wave is physically correct. A demonstration of this phenomenon in a simple model I owe to J. C. Martin.

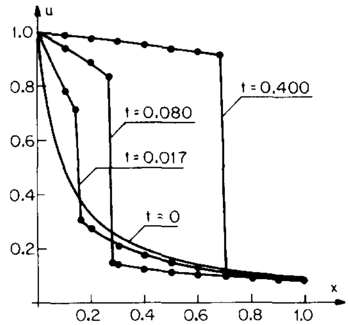

FIG. 3. The development of a sharp front with time t from the smooth initial distribution $u(x, 0) = 0.1/(0.1 + x)$.

4. Numerical Experiments

The numerical experiments were carried out in March 1979 on the DEK 10 computer at the University of Southern California. The reported results were obtained with the space increment $\Delta x = 0.02$ and the time step Δt such that the CFL condition was satisfied. The numerical results show that with the initial conditions (a) and (b) after 35 time steps with $\Delta t = 0.0034$ the discontinuity is kept perfectly sharp within one space step. The computed speed of propagation is $s = 1.014$ for (a), and $s_- = -3.38$ and $s_+ = 3.89$ for (b). The expansion waves connect the states between 0.955 and 1.00 for (a), and 0.305 and 0.494 for (b). In the next example the initial distribution is $u(x, 0) = 0.1/(0.1 + x)$ and the boundary condition is $u(0, t) = 1.0$ for $t \geqslant 0$. Here all the situations (with $u_L > u_R$) shown in Fig. 1 occur in solving the Riemann problems. The development of a sharp front as the time progresses is depicted on Fig. 3. In this example a variable time step was employed. The computed solution is drawn for the values of $t = 0$, 0.017, 0.080 and 0.400. Discontinuity of large magnitude develops much earlier that in the absence of gravity in agreement with the conclusion in Fayers and Sheldon [3], for the considered range of flow rates; see also Martin [5].

The number of the intermediate points u_M between u_L and u_R is less than or equal to the number of inflection points of $f(u)$ in the same interval. A general computer program for $f(u)$ with many inflection points could be quite complex. On the other hand the actual code for the present problem is only slightly more complicated than that for the simpler case reported in Concus and Proskurowski [1]. As a consequence, computational expenses remain practically the same. The CPU time for a one time step was 30 msec on the DEK 10.

Acknowledgments

I would like to thank R. P. Kendall for Ref. [2], F. R. Allen for Ref. [3], and J. C. Martin for Ref. [5].

REFERENCES

1. P. CONCUS AND W. PROSKUROWSKI, *J. Comput. Phys.* **30** (1979), 153–166.
2. J. DOUGLAS, B. L. DARLOW, M. WHEELER, AND R. P. KENDALL, *in* "Proceedings, Fifth SPE Symposium on Reservoir Simulation, Denver, Colorado, 1979," pp. 66–72.
3. F. J. FAYERS AND J. W. SHELDON, *Trans. AIME* **216** (1959), 147–156.
4. I. M. GELFAND, *Amer. Math. Soc. Transl. Ser. 2* **29** (1963), 295–381.
5. J. C. MARTIN, *Producers Monthly* **22** (April 1958).
6. G. STRANG, *SIAM J. Numer. Anal.* **5** (1968), 506–517.